Second Quantization

This note is an introduction to the topic of "second quantization", and hence to quantum "field theory". In the Electromagnetic Interactions note, we have already been exposed to these ideas in our quantization of the electromagnetic field in terms of photons. We develop the concepts more generally here, for both bosons and fermions. One of the uses of this new formalism is that it provides a powerful structure for dealing with the symmetries of the states and operators for systems with many identical particles.

Bosons

Suppose we have an n (identical) boson system, where all n bosons are in the lowest, ϕ_0, level. Denote this state by $|n\rangle$.

We assume that $|n\rangle$ is normalized: $\langle n|\, n\rangle = 1$. Since the particles are bosons, we can have $n = 0, 1, 2, \ldots$, where $|0\rangle$ is the state with no particles (referred to as the "vacuum").

We begin with the idea that emerged in our quantization of the electromagnetic field, and introduce operators that add or remove particles from a system, similar to the changing of excitation quanta of a harmonic oscillator.

Now define "annihilation" (or "destruction") operators as in case of harmonic oscillator:

$$a_0 |n\rangle = \sqrt{n}\, |n-1\rangle \tag{1}$$

i.e. the operator a_0 annihilates a single particle from the state $|n\rangle$ and produce a corresponding $n-1$ particles state.

An equivalent adjoint operator called creation operator can be defined as

$$a_0^\dagger |n-1\rangle = \sqrt{n}\, |n\rangle\,.$$

The operator $a_0^\dagger$ acts on $n-1$ particle state producing original state $|n\rangle$ Let us consider an operator $\hat{N}_0$ such that

$$\hat{N}_0 = a_0^\dagger a_0$$

Then,

$$\begin{aligned}
\hat{N}_0 |n\rangle &= a_0^\dagger a_0 |n\rangle \\
&= a_0^\dagger \sqrt{n}\, |n-1\rangle \\
&= \sqrt{n} a_0^\dagger |n-1\rangle \\
&= \sqrt{n}\sqrt{n}\, |n\rangle \\
&= n\, |n\rangle
\end{aligned}$$

$$\therefore \boxed{\hat{N}_0 |n\rangle = n\, |n\rangle} \tag{2}$$

Therefore, $\hat{N}_0$ is a Hermitian operator that counts the number of particles in state $|n\rangle$ and is called number operator.

We may write the n-particle state in terms of the vacuum state by:

$$\boxed{|n\rangle = \frac{(a_0^\dagger)^n}{\sqrt{n!}}\, |0\rangle}. \tag{3}$$

The operator a_0 and $a_0^\dagger$ obey commutation relation:

$$\left[a_0, a_0^\dagger\right] = 1$$

Let us generalize the above definitions to a Boson state of the form, $|n_0, n_1, n_2, \cdots\rangle$ where n_i represents the occupation number for the boson state i.

$$a_l \left|\cdots, n_l, \cdots\right\rangle = \sqrt{n_l}\left|\cdots, n_l - 1, \cdots\right\rangle \tag{4}$$

$$a_l a_j \left|\cdots, n_l, \cdots, n_j, \cdots\right\rangle = \sqrt{n_l}\sqrt{n_j}\left|\cdots, n_l - 1, \cdots n_j - 1, \cdots\right\rangle \tag{5}$$

The generalized commutation relations are:

$$\left[a_j, a_l^\dagger\right] = \delta_{j,l} \tag{6}$$

$$\left[a_j, a_l\right] = \left[a_j^\dagger, a_l^\dagger\right] = 0 \tag{7}$$

Since Bosons can commute, operations of the type $a_i^\dagger a_j^\dagger \left|\cdots, n_i, \cdots, n_j, \cdots\right\rangle$ give the same result irrespective of the order of operators $a_i^\dagger$ and $a_j^\dagger$ which is not in case with fermions.

Consider two levels, ϕ_0 and ϕ_1. Let $|n_0, n_1\rangle$ be the state with n_0 bosons in ϕ_0 and n_1 bosons in ϕ_1. As before, define,

$$a_0 |n_0, n_1\rangle = \sqrt{n_0}\,|n_0 - 1, n_1\rangle \tag{8}$$

$$a_0^\dagger |n_0, n_1\rangle = \sqrt{n_0 + 1}\,|n_0 + 1, n_1\rangle, \tag{9}$$

and also,

$$a_1 |n_0, n_1\rangle = \sqrt{n_1}\,|n_0, n_1 - 1\rangle \tag{10}$$

$$a_1^\dagger |n_0, n_1\rangle = \sqrt{n_1 + 1}\,|n_0, n_1 + 1\rangle. \tag{11}$$

In addition to the earlier commutation relations, we have that the annihilation and creation operators for different levels commute with each other:

$$[a_0, a_1] = 0; \qquad [a_0^\dagger, a_1] = 0 \tag{12}$$

$$\left[a_0, a_1^\dagger\right] = 0; \qquad [a_0^\dagger, a_1^\dagger] = 0 \tag{13}$$

We can construct an arbitrary state from the vacuum by:

$$|n_0, n_1\rangle = \frac{(a_0^\dagger)^{n_0}}{\sqrt{n_0!}}\frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}}|0, 0\rangle \tag{14}$$

The total number operator is now

$$\hat{N} = \hat{N}_0 + \hat{N}_1 = a_0^\dagger a_0 + a_1^\dagger a_1 \tag{15}$$

so that

$$\hat{N}\,|n_0, n_1\rangle = (n_0 + n_1)\,|n_0, n_1\rangle. \tag{16}$$

We may generalize these results to spaces with an arbitrary number of single particle states. Thus, let

$|n_0, n_1, \ldots\rangle$ be a vector in such a space. For the case of bosons, we have, in general:

$$[a_i, a_j^\dagger] = \delta_{ij}, \tag{17}$$

$$\left[a_i, a_j\right] = [a_i^\dagger, a_j^\dagger] = 0, \tag{18}$$

$$|n_0, n_1, \ldots\rangle = \cdots \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_0^\dagger)^{n_0}}{\sqrt{n_0!}} |0\rangle, \tag{19}$$

where $|0\rangle$ represents the vacuum state, with all $n_i = 0$.