## Fermions

Let us assume that the particles occupying the state $|n_0\rangle$ are fermions. We are restricted by Pauli's exclusion principle that $n_0 = 0, 1$.

Now, in case of fermions, we can define creation and annihilation operators such that

$$a_0 |1\rangle = |0\rangle, \qquad a_0 |0\rangle = 0; \tag{1}$$

$$a_0^\dagger |1\rangle = 0, \qquad a_0^\dagger |0\rangle = |1\rangle. \tag{2}$$

In the $|0\rangle, |1\rangle$ basis, these operators are the $2 \times 2$ matrices:

$$a_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$a_0^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

With this explicit representation, we see that they are hermitian conjugate to each other. The algebraic properties of the fermion operators are different from those of the boson operators. The commutator, in the $|0\rangle, |1\rangle$ basis, is

$$[a_0, a_0^\dagger] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq I \tag{3}$$

By construction, we cannot put two fermions in the same state with these operators.

Consider the anticommutator:

$$\{a_0, a_0^\dagger\} |1\rangle = (a_0 a_0^\dagger + a_0^\dagger a_0) |1\rangle = |1\rangle, \tag{4}$$

$$\{a_0, a_0^\dagger\} |0\rangle = |0\rangle \tag{5}$$

That is, $\{a_0, a_0^\dagger\} = 1$. Also,

$$\{a_0, a_0\} = 0, \tag{6}$$

$$\{a_0^\dagger, a_0^\dagger\} = 0. \tag{7}$$

The number of particles operator is $N_0 = a_0^\dagger a_0$.

In the case of fermions, we now have four possible states: $|0,0\rangle, |1,0\rangle, |0,1\rangle$, and $|1,1\rangle$. We define:

$$a_0^\dagger |0,0\rangle = |1,0\rangle; \quad a_0^\dagger |1,0\rangle = 0, \tag{8}$$

$$a_0 |1,0\rangle = |0,0\rangle; \quad a_0 |0,0\rangle = 0, \tag{9}$$

$$a_0 |0,1\rangle = 0; \quad a_0^\dagger |1,1\rangle = 0, \tag{10}$$

$$a_1^\dagger |0,0\rangle = |0,1\rangle; \quad a_1 |0,0\rangle = a_1 |1,0\rangle = 0, \tag{11}$$

$$a_1^\dagger |1,0\rangle = |1,1\rangle; \quad a_1 |0,1\rangle = |0,0\rangle, \tag{12}$$

$$a_1^\dagger |0,1\rangle = a_1^\dagger |1,1\rangle = 0; \quad a_1 |1,1\rangle = |1,0\rangle. \tag{13}$$

But we must be careful in writing down the remaining actions, of $a_0, a_0^\dagger$ on the states with $n_1 = 1$. These actions are constrained by consistency with the exclusion principle. We must get a sign change if we interchange the two fermions in a state. Thus, consider using the $a$ and $a^\dagger$ operators to "interchange" the two fermions in the $|1,1\rangle$ state: First, take the fermion away from $\phi_1$,

$$|1,1\rangle \to |1,0\rangle = a_1 |1,1\rangle . \tag{14}$$

Then "move" the other fermion from $\phi_0$ to $\phi_1$,

$$|1,0\rangle \to |0,1\rangle = a_1^\dagger a_0 |1,0\rangle . \tag{15}$$

Finally, restore the other one to $\phi_0$,

$$|0,1\rangle \to a_0^\dagger |0,1\rangle = a_0^\dagger a_1^\dagger a_0 a_1 |1,1\rangle \tag{16}$$

We require the result to be a sign change, i.e.,

$$a_0^\dagger |0,1\rangle = -|1,1\rangle . \tag{17}$$

Since $a_0$ is the hermitian conjugate of $a_0^\dagger$, we also have $a_0 |1,1\rangle = -|0,1\rangle$.

We therefore have the anticommutation relations:

$$\{a_0, a_0^\dagger\} = \{a_1, a_1^\dagger\} = 1. \tag{18}$$

All other anticommutators are zero, including $\{a_0, a_1\} = \{a_0, a_1^\dagger\} = 0$, following from the antisymmetry of fermion states under interchange.

For the fermion case, we have the generalization:

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \tag{19}$$

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0, \tag{20}$$

$$|n_0, n_1, \ldots\rangle = \cdots (a_1^\dagger)^{n_1} (a_0^\dagger)^{n_0} |0\rangle . \tag{21}$$

The general relation for a state $|n_0, n_1, \cdots\rangle$ with creation and annihilation operator is given by

$$a_i^\dagger |\cdots, n_{i-1}, n_i, n_{i+1}, \cdots\rangle = (-1)^{\eta_i} |\cdots, n_{i-1}, 1, n_{i+1}, \cdots\rangle \quad \text{for } n_i = 0 \tag{22}$$

$$= 0 \quad \text{for } n_i = 1 \tag{23}$$

$$a_i |\cdots, n_{i-1}, n_i, n_{i+1}, \cdots\rangle = (-1)^{\eta_i} |\cdots, n_{i-1}, 0, n_{i+1}, \cdots\rangle \quad \text{for } n_i = 1 \tag{24}$$

$$= 0 \quad \text{for } n_i = 0 \tag{25}$$

where $\eta_i$ is the number of occupied state to the left of state $i$.

Let $\Psi$ be an occupation number wave function that contains the state $i$ but not the state $j$. Let us assume $i > j$. Also $\eta_i$ and $\eta_j$ represent the number of occupied state immediately to the left of $i$ and $j$ respectively. Then ,

$$a_j^\dagger a_i \Psi = (-1)^{\eta_i + \eta_j} \left|\cdots, n_{j-1}, 1, \cdots, n_{i-1}, 0, \cdots\right\rangle \tag{26}$$

$$a_i a_j^\dagger \Psi = (-1)^{\eta_i + \eta_j + 1} \left|\cdots, n_{j-1}, 1, \cdots, n_{i-1}, 0, \cdots\right\rangle \tag{27}$$

$$\therefore \left[a_j^\dagger a_i + a_i a_j^\dagger\right]\Psi = (-1)^{\eta_i+\eta_j}[1-1]\left|\cdots, n_{j-1}, 1, \cdots, n_{i-1}, 0, \cdots\right\rangle \tag{28}$$

Or equivalently,

$$\left[a_j^\dagger, a_i\right]_+ = a_j^\dagger a_i + a_i a_j^\dagger = 0 \quad \text{for } i \neq j \tag{29}$$

It is straightforward to show that, if $i = j$, $a_i^\dagger a_i + a_i a_i^\dagger = 1$. The general anticommutation rule is therefore,

$$\left[a_i^\dagger, a_j\right]_+ = \delta_{ij} \tag{30}$$

The number operator $\hat{N}$ can therefore be defined as

$$\hat{N} = \sum_i a_i^\dagger a_i \tag{31}$$

**Homework-01**

1. Show the following commutation relation of annihilation and creation operator for bosons.
   $[a_0\ a_1] = 0 \qquad [a_0^\dagger\ a_1] = 0$
   $[a_0\ a_1^\dagger] = 0 \qquad [a_0^\dagger\ a_1^\dagger] = 0$

2. Show that $\hat{N}\left|n_0, n_1\right\rangle = (n_0 + n_1)\left|n_0, n_1\right\rangle$ from the definition of $a_0$ and $a_1$ for bosons.

3. Show that $2 \times 2$ matrix

   $a_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $a_0^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ on the basis of $\left|0\right\rangle, \left|1\right\rangle$

4. Show that $[a_0, a_0^\dagger] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq I$

**Homework-02**

1. Show that $\left[a_0\ a_0^\dagger\right] = \left[a_1\ a_1^\dagger\right] = 1$ and $[a_0\ a_1] = \left[a_1\ a_1^\dagger\right] = 0$

2. Verify: $\left|n_0, n_1, .....\right\rangle = \left(a_1^\dagger\right)^{n_1}\left(a_0^\dagger\right)^{n_0}\left|0\right\rangle$ for five single particle state.

3. Show the following relations:

$$a_j^\dagger a_i \Psi = (-1)^{\eta_i+\eta_j}\left|\cdots, n_{j-1}, 1, \cdots, n_{i-1}, 0, \cdots\right\rangle$$
$$a_i a_j^\dagger \Psi = (-1)^{\eta_i+\eta_j+1}\left|\cdots, n_{j-1}, 1, \cdots, n_{i-1}, 0, \cdots\right\rangle$$
$$\left[a_j^\dagger a_i + a_i a_j^\dagger\right]\Psi = (-1)^{\eta_i+\eta_j}[1-1]\left|\cdots, n_{j-1}, 1, \cdots, n_{i-1}, 0, \cdots\right\rangle$$

   where $\eta_i$ and $\eta_j$ are number of occupied state.

4. Show that

$$\hat{N} = \sum_i a_i^\dagger a_i$$

   is a number operator in fermions.