1 Plasma Oscillation

We have developed a theory for the screening effects in an electron gas that assumes slow motion of the electrons. In general this is not true, especially at large-wave vector comparable to the inverse interparticle spacing. In this limit, collective excitations of the electron gas become accessible. Such collective excitations are termed *plasmons*.

The existence of plasma excitations can be established straight-forwardly from the equations of motion for the electron density in momentum space.To proceed, we recall the form of the Coulomb interaction

$$\sum_{\mathbf{k}} \frac{4\pi}{k^2} e^{ik\cdot(r_i - r_j)} = \frac{1}{\mid r_i - r_j \mid} \tag{1}$$

in momentum space, which implies that we can rewrite the Hamiltonian for our electron gas as

$$\begin{aligned} H &= \sum_i \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i\neq j}\sum_{\mathbf{k}} \frac{4\pi e^2}{k^2} e^{i\mathbf{k}\cdot(r_i - r_j)} \\ &= \sum_i \frac{p_i^2}{2m} + \sum_{\mathbf{k}} \frac{V_{\mathbf{k}}}{2}(n_{\mathbf{k}} n_{-\mathbf{k}} - N) \end{aligned} \tag{2}$$

with $n_k = \sum_i e^{ik\cdot r_i}$, N the number of electrons, and $V_{\mathbf{k}} = 4\pi e^2/k^2$. As we chose our box to be of unit volume, $n_{\mathbf{k}}$ appears without the V^{-1} normalization.

To determine the collective excitations, we need the time evolution of $n_{\mathbf{k}}$,

$$\begin{aligned} i\dot{n}_{\mathbf{k}} &= \left[n_{\mathbf{k}}, H\right]/\hbar \\ &= \frac{1}{\hbar}\sum_i \left[\frac{\hbar^2}{m}\nabla_{r_i} e^{ik\cdot r_i}\cdot\nabla_{r_i} + \frac{\hbar^2}{2m}\nabla_{r_i}^2 e^{ik\cdot r_i}\right] \\ &= -\sum_i e^{ik\cdot r_i}\left(\frac{k\cdot p_i}{m} + \frac{\hbar k^2}{2m}\right). \end{aligned} \tag{3}$$

The second derivative is

$$\begin{aligned}\ddot{n}_{\mathbf{k}} &= \sum_i e^{ik\cdot r_i}\left(\frac{k\cdot p_i}{m}+\frac{\hbar k^2}{2m}\right)^2 \\ &+ \frac{2\pi e^2}{\hbar}\sum_i\left[e^{ik\cdot r_i}\frac{k\cdot p_i}{m},\sum_q\left(\frac{n_{\mathbf{q}}^{\dagger}n_{\mathbf{q}}-N}{q^2}\right)\right] \\ &= \sum_i e^{ik.r_i}\left(\frac{k.p_i}{m}+\frac{\hbar k^2}{2m}\right)^2-\frac{4\pi e^2}{m}\sum_q k.q\frac{n_{k-q}\;\,n_q}{q^2}.\end{aligned} \tag{4}$$

We separate the $\mathbf{k}=\mathbf{q}$ interaction term, which can be written as,

$$4\pi e^2\frac{k^2}{mk^2}n_0 n_{\mathbf{k}} = \frac{4\pi e^2 n_e}{m}n_{\mathbf{k}} = \omega_p^2 n_{\mathbf{k}} \tag{5}$$

That ω_p represents the frequency of the collective oscillations of the electron gas can be seen by writing $\ddot{n}_{\mathbf{k}}$ in the suggestive form

$$\ddot{n}_{\mathbf{k}}+\omega_p^2 n_{\mathbf{k}} = \sum_i e^{ik\cdot r_i}\left(\frac{k\cdot p_i}{m}+\frac{\hbar k^2}{2m}\right)^2-\frac{4\pi e^2}{m}\sum_{q\neq k}k.q\frac{n_{k-q}\;\,n_q}{q^2}. \tag{6}$$

We see, then, that the density $n_{\mathbf{k}}$ oscillates at the frequency ω_p, the plasma frequency, if the terms on the right-hand side of Eq. (6) are small. The first term is of order $k^2 v_F^2 n_{\mathbf{k}}$. In the next term, a product of $n_{\mathbf{q}}$'s appears. Because the density is a sum of exponential terms with randomly varying phases, one might except that the contribution from the second term is minimal. The approximation that ignores this contribution is known as the *random-phase approximation*. A well-defined plasma frequency exists at this level of theory if

$$\omega_p^2 \gg k^2 v_F^2 \tag{7}$$

For a density of $n_e \sim 10^{23}\, e^-/cm^3$, $\omega_p \sim 10^{16} s^{-1}$, or, equivalently, the energy in a plasma oscillation is

$$\hbar\omega_p \sim 12eV \tag{8}$$

Such a high-energy excitation cannot be created by thermal or phonon-like oscillations of the ions. They also cannot be excited by a single electron. Plasma oscillations or plasmons arise from a collective motion of all the electrons in a solid. As such, plasmons are long-wavelength oscillations.We estimate the maximum wave vector for which plasmons exist by considering the ratio ω_p/κ_{TF}. Recall that $\kappa_{TF}^2 = 4me^2 p_F/\pi\hbar^3$. As a cosequence,

$$\omega_p/\kappa_{TF} = \left(\frac{4\pi e^2 n_e}{m}\frac{\pi\hbar^3}{4p_F m e^2}\right)^{\frac{1}{2}} \\ = \frac{p_F}{\sqrt{3}m} = \frac{v_F}{\sqrt{3}} \tag{9}$$

We find that $\omega_p \propto v_F\kappa_{TF}$. Comparison with Eq.(7) reveals that well-defined plasma oscillations exist if $\kappa_{TF} \gg k$. For $k \leq \kappa_{TF}$, the electrons act individually. Note also that $\omega_p^2 \sim 1/m$. For an interacting system, we replace m by the effective mass, m^*. In an insulator, $m^* \to \infty$. Hence, $\omega_p \sim 0$ in an insulator. Similarly, in a conductor m^* is finite and, as a consequence, $\omega_p \neq 0$. Consequently, the magnitude of ω_p is a sensitive test for the insulator-metal transition in an interacting electron system.

A final observation on plasmons is that their dispersion relationship is fundamentally tied to the dimensionality of space. If the electrons are confined to a plane ($d = 2$) but the electric field lines are allowed to live in three-dimensional space, thereby making the Coulomb interaction the standard $\frac{1}{r}$ potential (see Problem 8.3, Philip Philips), there is no gap to excite plasmon excitations. In addition (see Problem 8.1, Philip Philips), the screening length is independent of density in 2d, at least at the level of Thomas-Fermi. Both of these effects illustrate how fundamentally different a 2d electron gas is from its 3d counterpart.

1.1 Dispersion of Light

An application in which the plasma frequency naturally appears is the propagation of transverse electromagnetic radiation in metals. Consider the Maxwell equation for the magnetic induction in the presence of a current density,

$$\nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{4\pi e\mathbf{j}}{c} \tag{10}$$

The time derivative of this equation is

$$\frac{1}{c}\frac{\partial}{\partial t}\nabla\times\mathbf{B} = \frac{4\pi e}{c^2}\frac{\partial\mathbf{j}}{\partial t} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{11}$$

But $\partial\mathbf{B}/\partial ct = -\nabla\times\mathbf{E}$ and $\partial\mathbf{j}/\partial t = en_e\mathbf{E}/m$. For a transverse E-field, $\nabla.\mathbf{E} = 0$. As a consequence,

$$-\nabla\times(\nabla\times\mathbf{E}) = -\nabla(\nabla.\mathbf{E}) + \nabla^2\mathbf{E} = \nabla^2\mathbf{E}. \tag{12}$$

The time derivative can now be written as

$$\left(-\frac{\partial^2}{\partial t^2} + c^2\nabla^2 - \omega_p^2\right)\mathbf{E} = 0 \tag{13}$$

The resultant dispersion relationship for light in a metal,

$$\omega^2 = c^2k^2 + \omega_p^2 \tag{14}$$

illustrates clearly that transverse electromagnetic radiation cannot penetrate a metal for frequencies less than the plasma frequency.

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