1 Stopping Power of a Plasma

When an electron is injected into a plasma with some incoming energy, it is expected to be slowed as a result of the Coulomb interactions with the electrons in the plasma. On these grounds, Bethe argued that the rate of energy loss of the injected electron should be proportional to $\left|V_{int}(\mathbf{p})\right|^2$, where V_{int} is the Coulomb interaction between the plasma and the electron. For a Coulomb potential, $V_{int} \sim p^{-2}$. Summing over all incoming momentum values, we find that the energy loss is given by

$$\frac{dE}{dt} \propto \int d\mathbf{p}p\left|V_{int}(\mathbf{p})\right|^2 \propto \int_0^{p_F} \frac{dp}{p}. \tag{1}$$

This integral is logarithmically divergent at the lower limit. As a result, this simple account produces a divergent energy loss, which is clearly incorrect. We see immediately that, for a screened interaction, the divergence at the lower limit would vanish, thereby making dE/dt finite. This is the primary failure of the Bethe approach. We will now formulate this problem in a rigorous way that gets around the Bethe divergence by including the effects of screening.

The physical problem at hand is that of a metal in some initial state $|i\rangle$ and an electron with initial momentum p impinging on a metal. Upon interacting with the metal, the electron will have a new momentum,$\mathbf{p} - \hbar\mathbf{k}$, and the metal will be in some new state$|f\rangle$. We assume the states of the metal form a complete orthonormal set$\langle n|m\rangle = \delta_{nm}$. At the level of Fermi's golden rule, the transition rate between the initial and final states is

$$W_{\mathbf{p},\ \mathbf{p}-\hbar\mathbf{k}} = \frac{2\pi}{\hbar} \sum_f |\ \langle f, \mathbf{p} - \hbar k|V_{int}|i, \mathbf{p}\rangle\ |^2\ \delta\Big(\frac{\mathbf{p}^2}{2m} + E_i - \frac{(\mathbf{p} - \hbar\mathbf{k})^2}{2m} - E_f\Big), \tag{2}$$

where E_i and E_f are the total energies of the states $|i\rangle$ and $|f\rangle$, respectively. The interaction energy

$$V_{int}(\mathbf{r}) = \int d\mathbf{r}'\ [n(\mathbf{r}') - n_e]\ \frac{e^2}{|\ \mathbf{r}' - \mathbf{r}\ |} \tag{3}$$

includes the ion as well as the electron Coulomb energy. The initial state of the electron is a plane wave of the form $|\mathbf{p}\rangle = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}\Big/\sqrt{V}$ and the final electron state is $|\mathbf{p} - \hbar\mathbf{k}\rangle = e^{i(\mathbf{p}-\hbar\mathbf{k}).\mathbf{r}/\hbar}\Big/\sqrt{V}$. With these states, we rewrite the matrix element in Eq.(2) as

$$\begin{aligned}\langle f, \mathbf{p} - \hbar\mathbf{k}|V_{int}|i, \mathbf{p}\rangle &= \int d\mathbf{r}d\mathbf{r}'\langle f|n(\mathbf{r}') - n_e|i\rangle\frac{e^{i\mathbf{k}.\mathbf{r}}}{V}\frac{e^2}{|\ \mathbf{r} - \mathbf{r}'} \\ &= \frac{4\pi e^2}{k^2}\int\frac{d\mathbf{r}}{V}e^{i\mathbf{k}.\mathbf{r}}\ \langle f|n(\mathbf{r})|i\rangle\,.\end{aligned} \tag{4}$$

Using the integral representation of the δ- function,

$$2\pi\hbar\ \delta(\hbar\omega + E_i - E_f) = \int_{-\infty}^{\infty} e^{i(\hbar\omega+E_i-E_f)t/\hbar}dt,$$

we recast the transition rate as

$$\begin{aligned} W_{\mathbf{p},\mathbf{p}-\hbar\mathbf{k}} &= \left(\frac{4\pi e^2}{V\hbar k^2}\right)^2 \int_{-\infty}^{\infty} e^{i(\hbar\omega+E_i-E_f)t/\hbar} dt \sum_{f\neq i} |\langle f| \int d\mathbf{r} n(\mathbf{r}) e^{i\mathbf{k}.\mathbf{r}} |i\rangle|^2 \\ &= \left(\frac{4\pi e^2}{V\hbar k^2}\right)^2 \int_{-\infty}^{\infty} dt\, e^{i\omega t} \sum_{f\neq i} \langle f| n(\mathbf{k}) |i\rangle \; \langle i| e^{iE_i \frac{t}{\hbar}} \; n(\mathbf{k}) \; e^{-iE_f \frac{t}{\hbar}} |f\rangle, \end{aligned} \tag{5}$$

with $\hbar\omega = \epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}-\hbar\mathbf{k}}$. In the interaction representation,

$$\widehat{n}(\mathbf{k}, t) = e^{iH_0 \frac{t}{\hbar}} n(\mathbf{k}) e^{iH_0 \frac{t}{\hbar}}. \tag{6}$$

Consequently,

$$W_{\mathbf{p},\ \mathbf{p}-\hbar\mathbf{k}} = \left(\frac{4\pi e^2}{V\hbar k^2}\right)^2 \int_{-\infty}^{\infty} e^{i\omega t}\, dt \; \sum_{f\neq i} \langle i| \widehat{n}(\mathbf{k}, t) |f\rangle \langle f| n(\mathbf{k}) |i\rangle, \tag{7}$$

which can be simplified to

$$W_{\mathbf{p},\ \mathbf{p}-\hbar\mathbf{k}} = \left(\frac{4\pi e^2}{\hbar k^2}\right)^2 \frac{n_e}{V} \Big[S(\mathbf{k}, \omega) - \frac{1}{V} 2\pi\delta(\omega) n_e \Big] \tag{8}$$

using the definition of the structure function and the completeness relation for the metal states, $\sum |f\rangle \langle f| = 1$. As excepted, it is the dynamic structure factor that determines the response of our system to the incident electron.Screening effects are implicitly included in $S(\mathbf{k}, \omega)$. The Bethe result arises from the zero-frequency part of the transition rate.

We are primarily interested in the rate of energy loss to the plasma. This is determined by summing over all of the energy differences, $\epsilon - \epsilon_{\mathbf{p}-\hbar\mathbf{k}}$, weighted by the transition rate, W:

$$\begin{aligned} \frac{dE}{dt} &= -\sum_{\mathbf{k}} (\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}-\hbar\mathbf{k}}) W_{\mathbf{p},\ \mathbf{p}-\hbar\mathbf{k}} \\ &= -\frac{\hbar n_e}{V} \sum_{\mathbf{k}} \omega \left(\frac{4\pi e^2}{\hbar k^2}\right)^2 S(\mathbf{k}, \omega) \\ &= -n_e \int \frac{d\mathbf{k}}{(2\pi)^3} \left(\frac{4\pi e^2}{\hbar k^2}\right)^2 \int_{-\infty}^{\infty} \omega d\omega S(\mathbf{k}, \omega) \times \delta(\hbar\omega - \epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}-\hbar\mathbf{k}}). \end{aligned} \tag{9}$$

To evaluate this quantity, we switch to polar coordinates and perform first the θ integral for the angle between $\mathbf{p}$ and $\mathbf{p} - \hbar\mathbf{k}$:

$$\begin{aligned} 2\pi \int_0^{\pi} d\theta \sin\theta \delta(\hbar\omega - \epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}-\hbar\mathbf{k}}) &= 2\pi \int_{-1}^{1} dx \delta\Big(\hbar\omega + \frac{\hbar^2 k^2}{2m} - \frac{p\hbar k x}{m}\Big) \\ &= \frac{2\pi m}{p\hbar k} \Theta(k v_p - |\, \omega + \hbar k^2/2m \,|) \end{aligned} \tag{10}$$

With this result, the rate of energy loss to the plasma simplifies to

$$\frac{1}{v_p}\frac{dE}{dt} = \frac{-4e^4 n_e}{\hbar^2 v_p^2}\int_0^{\infty}\frac{dk}{k^3}\int_{-kv_p-\frac{\hbar k^2}{2m}}^{kv_p-\frac{\hbar k^2}{2m}} \omega d\omega S(\mathbf{k},\omega), \tag{11}$$

where v_p is the incoming velocity of the incident electron.

Complete stoppage of the electron by the plasma most likely to occur if the electron gas acts collectively, that is, if plasma oscillations dominate. Thus, the electron gas obtains maximum stopping power if $|\ \omega\ |= \omega_p$, the plasma frequency. We seek, then, an expression for the structure function in the limit of high frequency. From the definition of the dielectric function(see Eqs. **??** and **??**), we express the imaginary part of $\epsilon(\mathbf{k},\omega)^{-1}$,

$$Im\epsilon^{-1} = -\frac{2\pi e^2 n_e}{\hbar k^2}\left(1 - e^{-\beta\hbar\omega}\right) S(\mathbf{k},\omega), \tag{12}$$

in terms of the structure function and use the high-frequency expansion for the dielectric function

$$\epsilon(\omega) \sim \lim_{\eta\to 0}\left(1 - \frac{\omega_p^2}{(\omega + i\eta)^2}\right). \tag{13}$$

In this limit, the imaginary part of ϵ^{-1},

$$\begin{aligned} Im\epsilon^{-1}(\omega) &= \lim_{\eta\to 0} Im\left[\frac{\omega^2}{(\omega + i\eta)^2 - \omega_p^2}\right] \\ &= -\frac{\pi\omega_p}{2}[\delta(\omega - \omega_p) - \delta(\omega + \omega_p)], \end{aligned} \tag{14}$$

is a sum of two δ functions at $\pm\omega_p$. With the aid of Eq.(12), we see immediately that in the $k \to 0$ limit,

$$S(\mathbf{k},\omega) = \frac{\hbar\pi k^2}{m}\frac{[\delta(\omega - \omega_p) - \delta(\omega + \omega_p)]}{1 - e^{-\beta\hbar\omega}}. \tag{15}$$

The ω-integral in Eq.(11) is now straightforward:

$$\int_{\omega l}^{\omega u} \omega d\omega S(\mathbf{k}, le\omega) = \frac{\hbar\pi k^2}{m}(1+g_p)\Theta(\omega_l \leq \omega_p \leq \omega_u) - \frac{\pi k^2}{m} g_p \Theta(\omega_l \leq -\omega_p \leq \omega_u), \tag{16}$$

where $\omega_l = -kv_p - \hbar k^2/2m$ and $\omega_u = kv_p - \hbar k^2/2m$. In evaluating this integral, we introduced $g_p = (e^{\beta\hbar\omega_p} - 1)^{-1}$, whhich determines the number of plasmons thermally excited at a temperature T. The energy loss is transformed to

$$\frac{1}{v_p}\frac{dE}{dt} = -\Big(\frac{\omega_p e}{\hbar v_p}\Big)^2 \int \frac{dk}{k}(1+g_p)\Theta\Big(\hbar k v_p - \frac{(\hbar k)^2}{2m} - \hbar\omega_p \geq 0\Big) - g_p\Theta\Big(\frac{(\hbar k)^2}{2m} - \hbar k v_p \leq \hbar\omega_p \leq \frac{(\hbar k)^2}{2m} + \hbar k v_p\Big). \tag{17}$$

The first term represents the energy loss upon the creation of a plasmon and the latter the energy transferred to the electron by a plasmon thermally excited in the medium. As $T \to 0$, the probability that a plasmon will be thermally excited vanishes. As a result, plasmons can be excited only by an impinging electron. In this limit, the energy loss takes on the simple form:

$$\begin{aligned} \frac{1}{v_p}\frac{dE}{dt} &= -\Big(\frac{\omega_p e}{\hbar v_p}\Big)^2 \int \frac{dk}{k}\Theta\Big(\hbar k v_p - \frac{\hbar^2 k^2}{2m} - \hbar\omega_p \geq 0\Big) \\ &= -\Big(\frac{\omega_p}{\hbar v_p}\Big)^2 \ln\frac{k_+}{k_-}, \end{aligned} \tag{18}$$

where $k_\pm$ are the solutions to

$$\hbar^2 k^2 - 2m\hbar k v_p + 2m\hbar\omega_p = 0, \tag{19}$$

or, equivalently,

$$\hbar k_\pm = p \pm \sqrt{p^2 - 2m\hbar\omega_p}. \tag{20}$$

For the incident electron to excite a plasmon, $\frac{p^2}{2m} \geq \omega_p$. If we expand $k_\pm$ in this limit, we find that

$$k_\pm = p \pm (p - m\hbar\omega_p/p) = \Big\{2p - m\hbar\omega_p/pm\hbar\omega_p/p\Big\}. \tag{21}$$

We expect an absence of collective plasmon oscillations if $k \leq\sim a^{-1}$ where a is the interparticle spacing. We should then cut off k_+ at $2p_F$.
As a consequence,

$$\frac{dE}{dt} = -\frac{\omega_p^2 e^2}{v_p^2} \ln \frac{2pp_F}{m\hbar\omega_p}, \tag{22}$$

which is completely well behaved and finite, unlike the Bethe result.

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