1 Ultrasonic Attenuation

Imagine that we send a beam of phonons into a metal. The total rate at which the phonons are absorbed is determined by the direct absorption into the metal and emission back into the beam. Consequently, the rate at which a beam loses $N_{\mathbf{q},\lambda}$ phonons per unit is given by a kinetic gain-loss equation:

$$\frac{\mathrm{d}N_{\mathbf{q},\lambda}}{\mathrm{d}t} = -\sum_{\mathbf{p}}(W^{abs}_{\mathbf{p}\to\mathbf{p}+\mathbf{q}} - W^{emis}_{\mathbf{p}\to\mathbf{p}-\mathbf{q}}). \tag{1}$$

The first term represents the absorption of phonons from the beam and the second term the reemission of phonons into the beam and the second term the reemission of phonons into the beam. Inclusion of the re-emission term is essential to describe the correct physics.

We can simplify our kinetic equation by calling that the structure function for free electrons is given by

$$n_e S_0(\mathbf{p},\omega) = \frac{2}{V}\sum_{\mathbf{p}'} f_{\mathbf{p}'}(1 - f_{\mathbf{p}+\mathbf{p}'})2\pi\hbar\delta(\hbar\omega - \epsilon_{\mathbf{p}+\mathbf{p}'} + \epsilon_{\mathbf{p}'}). \tag{2}$$

Physically, $S_0(\mathbf{p},\omega)$ is the density of electron-hole excitations separated by an energy $\hbar\omega$. Inspection of the expressions for the phonon absorption and emission rates reveals that they are directly proportional to the right-hand side of the equation for the structure function. Let us assume that the electronic states are perfect plane waves with free-particle energies $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m$. As a result, the matrix element in Eq.(**??** 13 in Lec-03) is equal to unity: $\alpha_{\mathbf{k}+\mathbf{q},\mathbf{q}} = 1$. Consequently, we rewrite the net absorption and emission rates as

$$\sum_{\mathbf{p}} W^{abs}_{\mathbf{p}\to\mathbf{p}+\mathbf{q}} = \frac{n_e V}{2\hbar^2}\langle M_{\mathbf{q}}\rangle^2 N_{\mathbf{q},\lambda} S_0(\mathbf{q},\hbar\omega_{\mathbf{q}}) \tag{3}$$

and

$$\sum_{\mathbf{p}} W^{emis}_{\mathbf{p}\to\mathbf{p}-\mathbf{q}} = \frac{n_e V}{2\hbar^2}\langle M_{\mathbf{q}}\rangle^2 (N_{\mathbf{q},\lambda}+1) S_0(-\mathbf{q},-\hbar\omega_{\mathbf{q}}). \tag{4}$$

In the context of the fluctuation-dissipation therorem, we showed that

$$S_0(\mathbf{p},\hbar\omega) = e^{\beta\hbar\omega} S_0(-\mathbf{p},-\hbar\omega). \tag{5}$$

Substitution of this result into the equation of motion for $N_{\mathbf{q},\lambda}(t)$ yields

$$\ddot{N}_{\mathbf{q},\lambda} = -\frac{n_e V}{2\hbar^2}\langle M_{\mathbf{q}}\rangle^2 S_0(\mathbf{q}, \omega_{\mathbf{q}}\hbar)[N_{\mathbf{q},\lambda} - e^{-\beta\hbar\omega_{\mathbf{q}}}(N_{\mathbf{q},\lambda} + 1)]. \tag{6}$$

We define the net rate of phonons absorbed to be

$$\frac{1}{\tau_{ph}} = \frac{n_e V}{2\hbar^2}\langle M_{\mathbf{q}}\rangle^2 (1 - e^{-\beta\hbar\omega_{\mathbf{q}}}) S_0(\mathbf{q}, \hbar\omega_{\mathbf{q}}) \tag{7}$$

and the equilibrium phonon distribution, $N^{eq}_{\mathbf{q},\lambda}$, to be the standard Bose-Einstein distribution function,

$$N^{eq}_{\mathbf{q},\lambda} = \frac{1}{e^{\beta\hbar\omega_{\mathbf{q}}} - 1}. \tag{8}$$

Consequently, our equations of motion become

$$\ddot{N}_{\mathbf{q},\lambda} = \frac{-1}{\tau_{ph}}[N_{\mathbf{q},\lambda} - N^{eq}_{\mathbf{q},\lambda}], \tag{9}$$

and the solution to this linear differential equationn has the characteristic

$$N_{\mathbf{q},\lambda}(t) = N^{eq}_{\mathbf{q},\lambda} + e^{-t/\tau_{ph}}(N_{\mathbf{q},\lambda}(t=0) - N^{eq}_{\mathbf{q},\lambda}) \tag{10}$$

exponential form. We find, then, that the number of phonons absorbed relaxes to an equilibrium value at long times with a rate $1/\tau_{ph}$. This effect is known as *ultrasonic attentuation*, the loss of phonons to a medium as a result of interactions with electrons. Because electrons in a superconductor are bound together in pairs, with a binding energy proportional to the gap, they can absorb phonons only if the phonon frequency exceeds a critical value. As a consequence, ultrasound attenuation is used as a tool for measuring the gap in a superconductor.

A final observation is that the calculation we have performed here is valid only if electron interactions are negligible. That is, if τ_{ee} is the effective time scale for electron scattering, then our calculation is valid if $\omega_{\mathbf{q}}\tau_{ee} \gg 1$. If this condition does not hold and $\tau_{ee}\omega_{\mathbf{q}} < 1$, then sound waves are attenuated via electron scattering rather than by phonon-induced electron-hole pairs. For completeness, let us know evaluate τ_{ph}. At $T = 0$, we showed that at intermediate frequencies, $n_e S_0(\mathbf{k}, \omega) = m^2\omega/\pi k\hbar^2$. Because $\langle M_{\mathbf{q}}\rangle^2 \sim q^2 V_{et}(\mathbf{q})/\omega_q$, $1/\tau_{ph} \sim q\langle V_{ei}(\mathbf{q})\rangle^2$. Focusing only on the zero frequency part of the structure function, we find that $\epsilon_{\mathbf{p}} = \epsilon_{\mathbf{p+k}}$. For $p = p_F$, the transferred momentum is $k = 2p_F$. Here, at low frequencies, τ_{ph} is determined by particle scattering across the Fermi surface.