1   Electrical Conduction

The conductivity in a metal is measured by applying a voltage or an electric field to the material. The resultant current density

$$\mathbf{j} = \sigma \mathbf{E} \tag{1}$$

is directly proportional to the electric field through the conductivity,σ. The current density is in the direction of the carrier velocity,$\mathbf{v}$. The constant of proportionality is the net charge density, $-en_e$. Consequently,

$$\mathbf{j} = -n_e e \mathbf{v} \tag{2}$$

Let us now express $\mathbf{v}$ in terms of $\mathbf{E}$. In the absence of an electric field, $\langle \mathbf{v} \rangle = 0$ because $\mathbf{v}$ is randomized. Let τ represent the collision time of the electron. That is, over a time period τ, an electron is moving with constant velocity that we estimate as follows. The force exerted on an electron by the electric field is simply $-e\mathbf{E}$. The acceleration of the electron is,then, $-e\mathbf{E}/m$. If the acceleration is constant over a time τ, then the average velocity is $\mathbf{v}_{avg} = -e\mathbf{E}\tau/m$ and the current density is

$$\mathbf{j} = \frac{e^2 n_e}{m} \mathbf{E}\tau = \sigma \mathbf{E}, \tag{3}$$

or, equivalently,

$$\sigma = \frac{n_e e^2 \tau}{m}. \tag{4}$$

This is the Drude formula for the conductivity.

We have expressed the conductivity, then, in terms of a single unknown quantity, τ, the relaxation or collision time. Experimentally, once the conductivity is measured, τ can be extracted. Listed below are a few relaxation times for the alkali earth metals.

Element	77K	273K
Li	7.3×10^{-14}s	8.8×10^{-15}s
Na	1.7×10^{-13}s	3.2×10^{-14}s
K	1.8×10^{-13}s	4.1×10^{-14}s
Rb	1.4×10^{-13}s	2.8×10^{-14}s
Cs	8.6×10^{-14}s	2.1×10^{-14}s

We now want to develop a general theory that can account for the relaxation time and, hence, the conductivity. In a pure metal, the primary source of resistance is via interactions with lattice phonons. Any successful account of σ in a metal must explain the following: 1) σ is independent of E for moderate values of E, 2) The wide variation of σ from metal to metal, 3) The Wiedemann-Franz law that the ratio of $\kappa/\sigma = T$, where κ is the thermal conductivity, and 4) $\sigma \sim 1/T$ in most metals with a transition to $\sigma \sim t^{-5}$ at $T \to 0$. In this chapter, we focus entirely on the crossover from $1/T$ to T^{-5} behavior at low temperatures.

1.1   Boltzmann Equation

To proceed, we develop the Boltzmann transport theory. We introduce the distribution function $f(\mathbf{r}, \mathbf{k}, t)$, which defines the probability that a quantum "state" is occupied with momentum $\mathbf{k}$ and position $\mathbf{r}$ at time t. Although we are interested in only one band, f can be generalized to include all bands in a solid. The distribution function f specifies both the position and the momentum of an electron in a quantum state. Adopting such a distribution function is valid strictly at long wavelengths, that is, $\lambda \gg \hbar v_F / \kappa_B T$. Otherwise, the uncertainity principle is violated.

Consider the volume element $d\mathbf{k}d\mathbf{r}/(2\pi\hbar)^3 = d\Omega$. The product of this differential volume element with f, $fd\Omega$, defines the number of electrons in $d\Omega$. In the problem at hand, interactions with phonons alter the occupation in phase space. Let us refer to such processes as *lattice collisions*. Clearly, df/dt would be zero if no such collisions occured. In fact, for a solid in equilibrium, f is simplify the Fermi-Dirac distribution. For the non-equilibrium case, f must be determined from the general equations of motion. The total time derivative of f,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial t} + \dot{\mathbf{k}}.\boldsymbol{\nabla}_{\mathbf{k}}f + \dot{\mathbf{r}}.\boldsymbol{\nabla}_{\mathbf{r}}f = \frac{\partial f}{\partial t}\Big|_{coll}, \tag{5}$$

is determined by all the terms that either implicitly or explicitly depend on time. This is the Boltzmann equation. Because each volume element should be equivalent, the average number of electrons entering and leaving a volume element should be a constant. As a result, $\frac{\partial f}{\partial t} = 0$. In addition, our system is homogeneous, even in the presence of an electric field. As a consequence, the spatial derivative of f vanishes and

$$\frac{\partial \mathbf{k}}{\partial t}.\boldsymbol{\nabla}_{\mathbf{k}}f = \frac{\partial f}{\partial t}\Big|_{coll} \tag{6}$$

the steady-state Boltzmann equation, results. Physically, $\frac{\partial \mathbf{k}}{\partial t}$ is the force on the electrons in the Fermi sea. In an electric field,

$$\frac{\partial \mathbf{k}}{\partial t} = -e\mathbf{E}, \tag{7}$$

and, consequently, all the electrons are accelerated equally by the field.

To apply this equation to an electron-lattice problem, we must include an analogous Boltzmann equation is one in which the momentum term is absent, because in an unstrained crystal, there is no force on phonons. Let g be the phonon distribution function. It follows that

$$\frac{\partial g}{\partial t}\Big|_{coll} = \dot{\mathbf{r}}.\boldsymbol{\nabla}_{\mathbf{r}}g. \tag{8}$$

In all of our calculations to follow, we will replace g by its equilibrium value. Hence, we will not spend much time discussing g, though our treatment of f can be paralleled to solve for g.
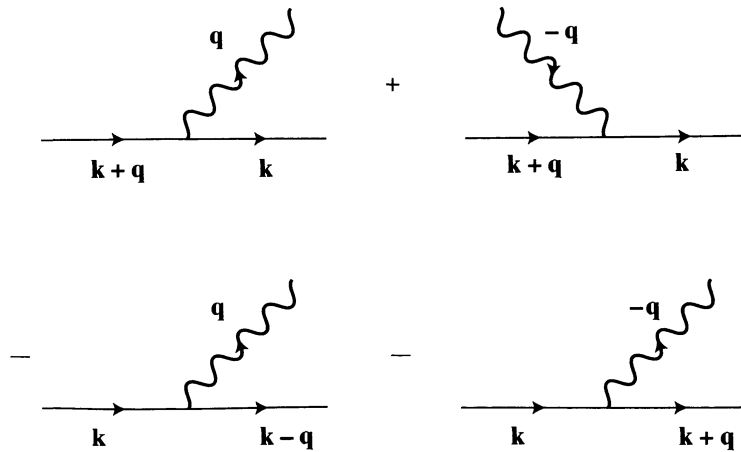
Figure 1: Electron-phonon collision terms that enter the Boltzmann equation.

Because f is determined by electron-phonon exchanges, the most general expression we can write for the collisions is one in which all possible electron-phonon processes are summed over. Let $\mathbf{W}^{e\mathbf{q}}$ represent an emission term and $\mathbf{W}^{a\mathbf{q}}$ a phonon absorption process with wave vector $\mathbf{q}$. The general collision terms that enter the Boltzmann equation are shown in Fig.(1) and can be written as a gain-loss master equation,

$$
\begin{aligned}
\frac{\partial f}{\partial t}\Big|_{coll} &= \sum_{\mathbf{q}} (W^{e\mathbf{q}}_{\mathbf{k+q}\to\mathbf{k}} + W^{a-\mathbf{q}}_{\mathbf{k+q}\to\mathbf{k}} - W^{a\mathbf{q}}_{\mathbf{k}\to\mathbf{k+q}} - W^{e-\mathbf{q}}_{\mathbf{k}\to\mathbf{k+q}}) \\
&= \sum_{\mathbf{q}} [gain(\mathbf{k}) - loss(\mathbf{k})],
\end{aligned}
\tag{9}
$$

for electron states with momentum $\mathbf{k}$ and $\mathbf{k+q}$. From the exact expressions for W in Eqs.(**??** 23 in Lec-03) and (**??** 24 in Lec-03), it is convenient to define

$$
W^0_{\mathbf{q}} = \frac{2\pi}{\hbar}\langle M_{\mathbf{q}}\rangle^2.
\tag{10}
$$

We have assumed that the electron states are plane waves. Hence, $\alpha_{\mathbf{k},\mathbf{q}} = 1$. The collision terms in Eq.(10) are easily computed if they are grouped as emission-absorption pairs:

$$\frac{\partial f}{\partial t}|_{coll} = \sum_{\mathbf{q}} \left(W^{e\mathbf{q}}_{\mathbf{k+q}\to\mathbf{k}} - W^{e-\mathbf{q}}_{\mathbf{k}\to\mathbf{k+q}} \right) - \left(W^{e-\mathbf{q}}_{\mathbf{k}\to\mathbf{k+q}} - W^{a-\mathbf{q}}_{\mathbf{k+q}\to\mathbf{k}} \right)$$

$$= \sum_{\mathbf{q}} W^0_{\mathbf{q}} \Big[(1 - f(\mathbf{k}))(f(\mathbf{k+q}))(g(\mathbf{q})+1)\delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k+q}} + \hbar\omega_{\mathbf{q}}) - (1 - f(\mathbf{k+q}))f(\mathbf{k})g(\mathbf{q})\delta(\epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k}} - \hbar\omega_{\mathbf{q}}) \Big]$$

$$+ W^0_{\mathbf{q}} \Big[(1 - f(\mathbf{k}))f(\mathbf{k+q})g(-\mathbf{q})\delta(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k+q}} - \hbar\omega_{\mathbf{q}}) \Big]$$

$$- (1 - f(\mathbf{k+q}))f(\mathbf{k})(g(-\mathbf{q})+1)\delta(\epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k}} + \hbar\omega_{\mathbf{q}}). \tag{11}$$

There are three common simplifications that are used to solve the Boltzmann equation for $f\mathbf{k}$. The first is to assume that $g(\mathbf{q}) = g_{equil}$, which is known as the *Bloch assumption*. The Bose-Einstein distribution is just g_{equil}. Let $N_{\mathbf{q}} = g_{equil} = N_{-\mathbf{q}}$. We also define

$$W_{\mathbf{q}}(\mathbf{k}, \mathbf{k}') = W^0_{\mathbf{q}}[\delta(\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}} + \hbar\omega_{\mathbf{q}})(N_{\mathbf{q}} + 1) + N_{\mathbf{q}}\delta(\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}} - \hbar\omega_{\mathbf{q}})]. \tag{12}$$

The right-hand side of the Boltzmann equation now simplifies to

$$\frac{\partial f}{\partial t}|_{coll} = \sum_{\mathbf{q}}[W_{\mathbf{q}}(\mathbf{k} + \mathbf{q}, \mathbf{k})f(\mathbf{k+q})(1 - f(\mathbf{k})) - W_{\mathbf{q}}(\mathbf{k}, \mathbf{k+q})f(\mathbf{k})(1 - f(\mathbf{k+q}))]. \tag{13}$$

1.2   Relaxation-Time Approximation

In the next step, we assume that on average, $f(\mathbf{k})$ is slowly varying when the field is applied. Collisions with phonons return the system to the equilibrium Fermi-Dirac distribution function, $f_0 = n_{\mathbf{k}}$. We write $f(\mathbf{k})$ as

$$f(\mathbf{k}) \equiv n_{\mathbf{k}} + \delta f_{\mathbf{k}}, \tag{14}$$

with $\delta f_{\mathbf{k}}$ is the variation of $f(\mathbf{k})$ induced by the electric field. We suspect that $\delta f_{\mathbf{k}}$ is proportional to the acceleration, $\delta\mathbf{k}/\delta t$. To see how this comes about, we make the ansatz that collision-induced changes of $f(\mathbf{k})$ relax the system back to $n_{\mathbf{k}}$ with a mean relaxation time $\tau(\mathbf{k})$, such that

$$\frac{\partial f}{\partial t}|_{coll} = -\frac{\delta f_{\mathbf{k}}}{\tau(\mathbf{k})}. \tag{15}$$

Note that introduction of a relaxation time at this stage can be done only at the expense of making τ **k**-dependent. Using Eqs. (6) and (7) and linearizing with respect to the fluctuation $\delta f_{\mathbf{k}}$, we find that

$$
\begin{aligned}
\frac{\partial f}{\partial t}\big|_{coll} = -\frac{\delta f_{\mathbf{k}}}{\tau} &= -eE.\nabla_{\mathbf{k}} f(\mathbf{k}) \\
&= -eE.\nabla_{k}\epsilon_{\mathbf{k}}\frac{\delta n_{\mathbf{k}}}{\delta \epsilon_{\mathbf{k}}} \\
&= e\frac{\mathbf{E}.\mathbf{k}}{m}n_{\mathbf{k}}(1-n_{\mathbf{k}})\beta,
\end{aligned}
\tag{16}
$$

where we have used the free-particle dispersion relation, $\epsilon_{\mathbf{k}} = \mathbf{k}^2/2m$. Because $\delta f_{\mathbf{k}} = f(\mathbf{k}) - n_{\mathbf{k}}$, we obtain

$$
f(\mathbf{k}) = n_{\mathbf{k}} + \beta\delta\Phi_{\mathbf{k}}n_{\mathbf{k}}(1-n_{\mathbf{k}}),
\tag{17}
$$

with

$$
\delta\Phi_{\mathbf{k}} = -e\frac{\mathbf{E}.\mathbf{k}}{m}\tau(\mathbf{k}).
\tag{18}
$$

We see, then, that once $\tau(\mathbf{k})$ is determined, we can find the distribution function $f(\mathbf{k})$ immediately and that the conductivity can be obtained through the Drude formula.

1.3 Low-Temperature Resistivity

To simplify the Boltzmann equation, we observe that $W_{\mathbf{q}}(\mathbf{k}, \mathbf{k}')$ obeys the symmetry relationship

$$
W_{\mathbf{q}}(\mathbf{k}, \mathbf{k}')e^{\beta\epsilon_{\mathbf{k}'}} = W_{\mathbf{q}}(\mathbf{k}, \mathbf{k}')e^{\beta\epsilon_{\mathbf{k}}}.
\tag{19}
$$

This statement is simply one of detailed balance. An equivalent, more useful way of writing Eq.(19) is

$$
W_{\mathbf{q}}(\mathbf{k}, \mathbf{k}') = e^{\beta(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'})}W_{\mathbf{q}}(\mathbf{k}', \mathbf{k})
\tag{20}
$$

$$
= \frac{n_{\mathbf{k}'(1-n_{\mathbf{k}})}}{n_{\mathbf{k}(1-n_{\mathbf{k}'})}}W_{\mathbf{q}}(\mathbf{k}', \mathbf{k}).
\tag{21}
$$

This identity implies that the quantity

$$Z_{\mathbf{q}}(\mathbf{k}, \mathbf{k}') = W_{\mathbf{q}}(\mathbf{k}, \mathbf{k}')n_{\mathbf{k}}(1 - n_{\mathbf{k}'}) \tag{22}$$

is symmetric with respect to interchange of its arguments. An immediate consequence of this identity is that the collision terms in the Boltzmann equation vanish identically, when $f(\mathbf{k}) = n_{\mathbf{k}}$:

$$W_{\mathbf{q}}(\mathbf{k}', \mathbf{k})(1 - n_{\mathbf{k}})n_{\mathbf{k}'} - W_{\mathbf{q}}(\mathbf{k}, \mathbf{k}')(1 - n_{\mathbf{k}'})n_{\mathbf{k}} = Z_{\mathbf{q}}(\mathbf{k}', \mathbf{k}) - Z_{\mathbf{q}}(\mathbf{k}, \mathbf{k}')$$
$$= 0. \tag{23}$$

The consequence of this identity are immediate. Recall that we have approximated $f(\mathbf{k})$ as $f(\mathbf{k}) \equiv n_{\mathbf{k}} + \beta n_{\mathbf{k}}(1 - n_{\mathbf{k}})\delta\Phi_{\mathbf{k}}$. Hence, only the terms with at least a linear variation of δf survive in the Boltzmann equation:

$$\frac{\partial f}{\partial t}|_{coll} = \sum_{\mathbf{q}} W_{\mathbf{q}}(\mathbf{k}', \mathbf{k})(\delta f_{\mathbf{k}'}(1 - n_{\mathbf{k}}) - n_{\mathbf{k}'}\delta f_{\mathbf{k}}) - \mathbf{k} \Leftrightarrow \mathbf{k}'$$
$$= \beta \sum_{\mathbf{q}} Z_{\mathbf{q}}(\mathbf{k}', \mathbf{k})[\delta\Phi_{\mathbf{k}'}(1 - n_{\mathbf{k}'}) - \delta\Phi_{\mathbf{k}}n_{\mathbf{k}} - \mathbf{k} \Leftrightarrow \mathbf{k}'] \tag{24}$$
$$= \beta \sum_{\mathbf{q}} Z_{\mathbf{q}}(\mathbf{k}', \mathbf{k})[\delta\Phi_{\mathbf{k}'} - \delta\Phi_{\mathbf{k}}].$$

In derivative Eq.(24), we dropped the $o(\delta\Phi^2)$ terms, thus obtaining the linearized Boltzmann equation.

As a result of the variation $\delta\Phi_{\mathbf{k}} \sim \tau(\mathbf{k})$, the Boltzmann equation is in general an integral equation that must be solved self-consistently by some ansatz. As in all integral equations, a variational principle applies, and we are guaranteed that a trial solution for $\delta\phi$ will result in a distribution function, f, that produces a higher energy than the true ground-state energy. In the relaxation-time approximation, $\delta\Phi_{\mathbf{k}} = -e\mathbf{E}.\mathbf{k}\tau(\mathbf{k}/m)$. In an electric field, the drift velocity is $\mathbf{v}_d = -e\mathbf{E}\tau/m$. As a consequence, $\delta\Phi_k \equiv \mathbf{v}_d.\mathbf{k}$ is known as the *drift-velocity ansatz*. Physically, this ansatz signifies that the electrons are in equilibrium with a drifting distribution. The drifting distribution is equivalent to the equilibrium Fermi-Dirac distribution with $\mathbf{q} \to \mathbf{q} - m\mathbf{v}_d$. As depicted in Figure, the drift-velocity ansatz amounts to an overall translation of the Fermi surface by an amount

proportional (in linear order) to $m\mathbf{v}_d$. We close the Boltzmann equation by averaging the collision terms over the electron momentum:

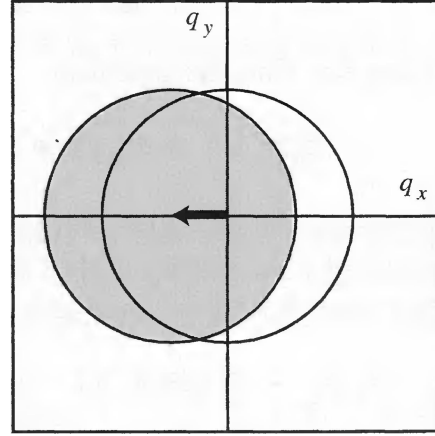Kapil Adhikari   *Prithvi Narayan Campus*

Figure 2: Equilibrium Fermi surface and the Fermi surface in the presence of a uniform electric field. The drift-velocity ansatz states that once the electric field is turned on, the new Fermi-Dirac distribution function is equivalent to the original one by simply translating $\vec{q} \rightarrow \vec{q} - m\vec{v}_d$.

$$
\begin{aligned}
\frac{\delta}{\delta t}\langle \mathbf{k}\rangle|_{coll} = \langle \dot{\mathbf{k}}\rangle &= 2\int \mathbf{k}\frac{\partial f}{\partial t}|_{coll}\frac{d\mathbf{k}}{(2\pi\hbar)^3} \\
&= -2\int \mathbf{k}\frac{d\mathbf{k}}{(2\pi\hbar)^3}\frac{f - f_0}{\tau} \\
&= -2\int \mathbf{k}\frac{d\mathbf{k}}{(2\pi\hbar)^3}\frac{f_0(\mathbf{k} - m\mathbf{v}_d)}{\tau} \\
&= -\frac{m}{\tau}\mathbf{v}_d n_e.
\end{aligned}
\tag{25}
$$

In deriving Eq. (25), we used the fact that $\langle \mathbf{k}\rangle$ in equilibrium vanishes.

Noting that the collision terms are anti-symmetric with respect to interchange of $\mathbf{k}$ and $\mathbf{k}$', we find upon substituting Eq. (25) into the Boltzmann equation that

$$
\begin{aligned}
\frac{-m\mathbf{v}_d n_e}{\tau} &= \frac{\beta}{V}\sum_{\mathbf{k}}\mathbf{k}\sum_{\mathbf{k}'}W_{\mathbf{q}}(\mathbf{k}, \mathbf{k}')n_{\mathbf{k}}(1 - n_{\mathbf{k}'})[\mathbf{v}_d.\mathbf{k}' - \mathbf{v}_d.\mathbf{k}] \\
&= \frac{\beta}{2V}\sum_{\mathbf{k}}\sum_{\mathbf{k}'}(\mathbf{k} - \mathbf{k}')W_{\mathbf{q}}(\mathbf{k}, \mathbf{k}')n_{\mathbf{k}}(1 - n_{\mathbf{k}'})[\mathbf{v}_d.\mathbf{k}' - \mathbf{v}_d.\mathbf{k}] \\
&= -\frac{\beta}{V}\sum_{\mathbf{k}}\sum_{\mathbf{q}}\mathbf{q}(\mathbf{v}_d.\mathbf{q})W_{\mathbf{q}}^0 n_{\mathbf{k}}(1 - n_{\mathbf{k+q}})[\delta(\epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k}} + \hbar\omega_{\mathbf{q}}) \times (N_{\mathbf{q}} + 1) + N_{\mathbf{q}}\delta(\epsilon_{\mathbf{k+q}} - \epsilon_{\mathbf{k}} - \hbar\omega_{\mathbf{q}})],
\end{aligned}
\tag{26}
$$

with $\mathbf{k}' = \mathbf{k} + \mathbf{q}$. Recalling the definition of the free-particle structure function (see Eq. **??** 2 in Lec-04), we reduce the linearized Boltzmann equation to

$$
\begin{aligned}
\frac{m\mathbf{v}_d n_e}{\tau} &= \frac{n_e\beta}{8\pi\hbar} \sum_q \mathbf{q}(\mathbf{v}_d.\mathbf{q}) W_{\mathbf{q}}^0 [S_0(\mathbf{q}, \hbar\omega_{\mathbf{q}}) N_{\mathbf{q}} + (N_{\mathbf{q}} + 1) S_0(\mathbf{q}, -\hbar\omega_{\mathbf{q}})] \\
&= \frac{n_e\beta}{4\pi\hbar} \sum_q \mathbf{q}(\mathbf{v}_d.\mathbf{q}) W_{\mathbf{q}}^0 N_{\mathbf{q}} S_0(\mathbf{q}, \hbar\omega_{\mathbf{q}}).
\end{aligned}
\tag{27}
$$

To simplify Eq. (27) further, we replace $\mathbf{q}(\mathbf{v}_d.\mathbf{q})$ with its angular average

$$
\langle \mathbf{q}(\mathbf{q}.\mathbf{v}_d) \rangle = \frac{1}{3} q^2 \mathbf{v}_d
\tag{28}
$$

Substitution of Eq. (28) into Eq. (27) results in the general expression

$$
\frac{1}{\tau} = \frac{\beta}{24\pi^2 M\hbar^3 m n_e n_{ion}} \int_0^{qD} \frac{q^6 dq}{\hbar\omega_{\mathbf{q}}} |V_{el}(q)|^2 \frac{S_0(q, \hbar\omega_{\mathbf{q}})}{e^{\beta\hbar\omega_{\mathbf{q}}} - 1}
\tag{29}
$$

for the relaxation time. In Eq. (29), q_D is the momentum cut-off on the phonon spectrum.

We need an expression for S_0 that captures the essential physics at low temperatures. We showed in the previous chapter that at T = 0, $S_0 \propto 1/q$. Asymptotically, this expression vanishes as $q \to \infty$ and, hence, is expected to be valid, as long as the phonon momentum is cut off. The Debye cut-off in the relaxation time justifies our use of the T = 0 limit. Away from T = 0, the explicit temperature dependence can be introduced by including the factor of (1 - $\exp(-\beta\hbar\omega_{\mathbf{q}})$), which appears in the original definition of $S_0(k, \omega)$. Consequently, we write

$$
S_0(q, \hbar\omega_{\mathbf{q}}) = \frac{m^2 \hbar\omega_{\mathbf{q}}}{\pi n_e \hbar^2 (1 - e^{-\beta\hbar\omega_{\mathbf{q}}})}.
\tag{30}
$$

We also need an expression for $V_{el}(q)$. In the Thomas-Fermi treatment of screening, we showed that

$$
V_{el}(q) = -\frac{4\pi Z e^2}{V(q^2 + \kappa_{TF}^2)}.
\tag{31}
$$

At low temperatures, we focus on the limit ($q \to 0$) of V_{el}. Hence, we approximate $V_{el}(q)$ with

$$
\begin{aligned}
V_{el}(q) &= -\frac{4\pi Z e^2}{V\kappa_{TF}^2} \\
&= -\frac{\pi^2 Z \hbar^3}{V m p_F} = -\frac{Z}{N(\epsilon_F)}
\end{aligned}
\tag{32}
$$

The relaxation time can be written as

$$\frac{1}{\tau} = \frac{Z^2 \beta m}{24\pi^3 n_e^2 \hbar^3 N^2(\epsilon_F) M n_{ion}} \int_0^{q_D} \frac{q^5 \, dq}{\left(1 - e^{-\beta \hbar \omega_q}\right)\left(e^{\beta \hbar \omega_q} - 1\right)} \tag{33}$$

For the phonon spectrum, we use the linear-dispersion relationship, $\omega_q = sq/\hbar$, where s is a constant. Let $x = \beta \omega_q$. With the observation that

$$\frac{1}{(e^x - 1)(1 - e^x)} = -\frac{\partial}{\partial x} \frac{1}{e^x - 1}, \tag{34}$$

we rewrite the relaxation time as

$$\frac{1}{\tau} = -\frac{Z^2 \beta m}{24\pi^3 \hbar^7 n_e^2 M N^2(\epsilon_F)(\beta s)^6} \int_0^{q_D s\hbar/k_B T} x^5 \frac{\partial}{\partial x} \frac{1}{(e^x - 1)} \tag{35}$$
$$= \alpha_0 T^5 J_5(T_D/T),$$

with

$$J_5(y) = -\int_0^y x^5 dx \frac{\partial}{\partial x} \frac{1}{e^x - 1}, \tag{36}$$

$T_D = q_D s\hbar/k_B$ is the Debye temperature, and

$$\alpha_0 = \frac{mZ^2 k_B^5}{\hbar^7 24\pi^3 n_e^2 n_{ion} M N^2(\epsilon_F) s^6} \tag{37}$$

There are two cases of interest. At low temperatures, $T_D/T \gg 1$, implying that the integral can be extended to infinity, leaving

$$J_5(\infty) = -\int_0^\infty x^5 dx \frac{\partial}{\partial x} \frac{1}{e^x - 1} \tag{38}$$
$$= 5!\zeta(5)$$

where $\zeta(n)$ is the Riemann-Zeta function. As a consequence,

$$\frac{1}{\tau} = 5!\zeta(5)\alpha_0 T^5 \tag{39}$$

for $T \ll T_D$. From the Drude formula, we have the resistivity $\rho \sim 1/\tau$. We see, then that for $T \ll T_D$, the resistivity scales as $\rho \sim T^5$. The origin of the T^5 contribution is as follows. A factor of T^3 arises from the number of phonons present at $T = 0$. The remaining factor T arise from momentum transfer and the fraction of electrons in the vicinity of T_F that can scatter. Each of these processes scales at T. Note the factor α_0 correctly represents the scaling of the resistivity in terms of the ion mass M, the electron density n_e, and the density of states $N(\epsilon_F)$.

Consider now the high-temperature limit. In this case, $T >> T_D$, and upper limit in eq (36) is $y << 1$. We can the series expand the integrand to obtain

$$J_5(y) = -\int_0^y x^5 dx \frac{\partial}{\partial x}\left(\frac{1}{x} + \cdots\right) = \int_0^y x^3 dx = \frac{y^4}{4} \tag{40}$$

Because $y = T_D/T$, we find that at high temperatures,

$$\rho \sim \frac{1}{\tau} \propto T^5 \left(\frac{T_D}{T}\right)^4 \sim T \tag{41}$$

Linear behavior sets in for $T/T_D > 0.2$. Of current interest is the linear-T resistivity in the normal state of the high-temperature copper oxide materials. As this behavior persists until the onset of superconductivity, the linear-T resistivity if of fundamentally different (and currently unknown) origin than the high-temperature linear-T resistivity that results from phonon scattering.

Of course, normal nonmagnetic impurities also contribyte to the resistivity. When the concentration (n_{imp}) of nonmagnetic impurities is small, the first Born approximation can be used. In this limit, impurity scattering contributes a constant term to the relaxation rate, proportional to ϵ_F at $T = 0$. This is a reflection that scattering at normal impurities of charge Z, the basis result for the impuroty relaxation rate is

$$\frac{1}{\tau_{imp}} \propto \frac{2n_{imp}Z^2\epsilon_F}{\hbar n_e} \tag{42}$$

Consequently, disorder in a metal is expected to lead to a nonzero resistance at $T = 0$, which is commonly referred to as the *residual resistance* in a metal. Of course, the situation changes dramatically in the strong-disorder regime. In this limit, perturbation theory breaks down. A transition to an Anderson localized state occurs when the strength of the disorder exceeds a critical value for $d > 2$ (see Chapter 12). In the localized regime, the electronic states decay exponentially with distance. For $d \leq 2$, the transition is particulary striking, as any amount of disorder leads to complete localization of all the electronic eigen-states. Localization of the eigenstates results in a vanishing of the dc-conductivity and the onset of an insulating state. In three dimensions, the disorder must exceed a critical value before insulating behaviour obtains. In an Anderson localized system, charge carriers myst be thermally excited if they are to transport at all. Consequently, activated transport typically obtains in insulators above $T = 0$.