1 Hydrodynamic Limit: Phonon Drag

We have been considering the Boltzmann equation in the context of the phonon contribution to the conductivity at low temperatures. In the course of the derivation, we assumed that the effective particle distribution in the presence of the electric field could be described by a drifting distributio, in which the electron momentum is replaced by $\vec{p} \to \vec{p} - m_e\vec{v}_d$. We treat here the situation in which long-wavelength and low-frequency variations dominate. This is the hydrodynamic limit:

$$\omega\tau_{ee} << 1 \tag{1}$$

$$kl << 1 \tag{2}$$

where ω and k are the frequency and wave vectors of the electron and τ_{ee} and l are collision times and mean-free paths, respectively.

We define the momentum density to be

$$g = 2\int \frac{d\vec{p}}{(2\pi\hbar)^3}\vec{p}f_{\vec{p}}(\vec{r}, t) \tag{3}$$

We are interested in the rate of change of momentum density. Specifically, our focus is on the rate at whihc the collison processes limit momentum exchange. Denoting the explicit time dependence of the momentum density in the time derivative as $\partial\vec{g}/\partial t$, we write the collision terms as

$$\begin{aligned}\left.\frac{\partial\vec{g}}{\partial t}\right|_{coll} &\equiv \frac{\partial\vec{g}}{\partial t} + 2\int \frac{d\vec{p}}{(2\pi\hbar)^3}\vec{p}\left(\dot{\vec{p}}\cdot\nabla_{\vec{p}}f_{\vec{p}} + \dot{\vec{r}}\cdot\nabla_{\vec{r}}f_{\vec{p}}\right)\\ &= \frac{\partial\vec{g}}{\partial t} + 2\int \frac{d\vec{p}}{(2\pi\hbar)^3}\vec{p}\left.\frac{\partial f_{\vec{p}}}{\partial t}\right|_{coll}\end{aligned} \tag{4}$$

We will find it expedient to rewrite the collision terms, using the definitions

$$\dot{p} = -\nabla_r\epsilon_{\vec{p}} \tag{5}$$

$$\dot{r} = \nabla_p\epsilon_p = v_p \tag{6}$$

in steady state. Note that in steady state, $\dot{f} = 0$. In the presence of an electric field, we replace the single-particle energy levels with $\epsilon_p \to \epsilon_p + e\Phi(\vec{r})$, where $\Phi(\vec{r})$ is the electrical potential. As a consequence, $\nabla_r\epsilon_p = -e\vec{E}$. If we substitute these definitions into equation (4), we obtain

$$\left.\frac{\partial\vec{g}}{\partial t}\right|_{coll} = \frac{\partial\vec{g}}{\partial t} + 2\int \frac{d\vec{p}}{(2\pi\hbar)^3}\vec{p}\left(\nabla_p\epsilon_p\cdot\nabla_r f_p - \nabla_r\epsilon_p\cdot\nabla_p f_p\right) \tag{7}$$

$$\Rightarrow \left.\frac{\partial\vec{g}}{\partial t}\right|_{coll} = \frac{\partial\vec{g}}{\partial t} + 2\int \frac{d\vec{p}}{(2\pi\hbar)^3}\left(\nabla_r\epsilon_p\right)f_p + 2\int \frac{d\vec{p}}{(2\pi\hbar)^3}\vec{p}\nabla_r\left(v_p f_p\right) \tag{8}$$

$$\Rightarrow \left.\frac{\partial \vec{g}}{\partial t}\right|_{coll} = \frac{\partial \vec{g}}{\partial t} - e\vec{E}\tilde{n}_e + \nabla_r \cdot T \tag{9}$$

where T is a stress tensor with components

$$T_{ij} = 2\int \frac{d\vec{p}}{(2\pi\hbar)^3} p_i \vec{v}_{p_j} f_{\vec{p}} \tag{10}$$

with $i, j = x, y, z$, and $\tilde{n}_e$ is the electron density in the presence of the electric field. In deriving equation (8) from equation (7), we integrated the last term in equation (7) by parts.

We compute T_{ij} assuming the electrons are in equillibrium with the drifting distribution created by an electric field. The distribution function fro a drifting velocity distribution is simply $f_p \to f_{p-mv_d}$. In evaluating $\partial\vec{g}/\partial t$, we translate $\vec{p} \to \vec{p} + m\vec{v}_d$; as a consequence, the distribution function will remain unchanged from the static Fermi-Dirac distribution. The stress tensor then becomes

$$T_{ij} = 2\int \frac{d\vec{p}}{(2\pi\hbar)^3} (p_i + mv_{d_i})\left(v_{p_j} + v_{d_j}\right) f_{\vec{p}} \tag{11}$$

$$= 2\int \frac{d\vec{p}}{(2\pi\hbar)^3} \left(p_i v_{p_j} + mv_{d_i} v_{d_j}\right) f_{\vec{p}} \tag{12}$$

because $\langle p \rangle = 0$. For an isotropic system, $\left\langle \vec{p}\vec{v}_p \right\rangle = \frac{1}{3}\left\langle \vec{p} \cdot \vec{v}_p \right\rangle$. In the context of the noninteracting electron gas, we showed that the pressure exerted by the electron gas is

$$P_e = \frac{2}{3}\int \frac{d\vec{p}}{(2\pi\hbar)^3} \vec{p} \cdot \vec{v} f_{\vec{p}} \tag{13}$$

As a consequence, the stress tensor reduces to

$$T_{ij} = \delta_{ij} P_e + mv_{d_i} v_{d_j} n_e \tag{14}$$

and the equation of motion for the momentum density

$$\left.\frac{\partial \vec{g}}{\partial t}\right|_{coll} = \frac{\partial \vec{g}}{\partial t} + \nabla P_e + \nabla \cdot (n_e m \vec{v}_d \vec{v}_d) - e\vec{E}\tilde{n}_e \tag{15}$$

contains the gradient of the pressure.

In the absence of collisions of any sort, momentum is conserved and the right-hand side of equation (15) vanishes, except for the electric field term. Consequently,

$$\left.\frac{\partial \vec{g}}{\partial t}\right|_{coll} + e\vec{E}\tilde{n}_e = 0 \tag{16}$$

is the basic equation underlying electrical transport.

When ions are present, we can replace the left-hand side of equation (15) with the relaxation-time approximation

$$\left.\frac{\partial\vec{g}}{\partial t}\right|_{coll} = -\frac{g}{\tau} = -\frac{\tilde{n}_e m\vec{v}_d}{\tau} \tag{17}$$

We expand the range of applicability of equation (17) even further by incorporating the ion velocity. Because the ions are moving, collisions with ions will cause equillibration to the ion velocity rather than to the drift velocity. If $\vec{u}_{ion}$ is the ion velocity, then

$$\left.\frac{\partial\vec{g}}{\partial t}\right|_{coll} = -\frac{\vec{g}_{eq}}{\tau} = -\tilde{n}_e m\left(\vec{v}_d - \vec{u}_{ion}\right) \tag{18}$$

Equations (15) to (18) constitute the hydrodynamic equations of motions for the electron momentum density. The suppression of the drift velocity by the velocity of the ions is known as *phonon drag*.