1 Sound Propagation

We can also use the hydrodynamic approach to understand the propagation of sound waves in a metal. A sound wave arises from the collective motion of ions in a crystal. For electrons, we showed that the frequency for collective oscillations is given by $\omega_p^2 = 4\pi n_e e^2/m$. Modifying this expression, so that the electric charge is replaces by the ion charge, $e \rightarrow eZ$ and $m \rightarrow M$ and $n_e \rightarrow Zn_i$, we find the equivalent ion "plasma" frequency is

$$\omega_{ion}^2 = \left(\frac{Z^3 m n_i}{n_e M}\right)\omega_p^2 \tag{1}$$

This conclusion is erroneous because the long-wavelength ion excitations should obey a linear k dispersion relationship, as opposed to the relatively constant dispersion relation for plasma oscillations.

To correct this problem, we must include both the electron and ion degrees of freedom in our hydrodynamic equations for the momentum density. If $\vec{g}_i on$ is the momentum density for the ions, then

$$\frac{\partial \vec{g}_{ion}}{\partial t} + eZ\vec{E}\tilde{n}_i = -\left(\frac{\partial \vec{g}_{ion}}{\partial t}\right)_{coll} \tag{2}$$

is the hydrodynamic equation for the ion degrees of freedom, with $\tilde{n}_i$ the ion density in the presence of the electric field. Because the charge on the ions is Ze, the coupling to the electric field has the opposite sign than in the electron problem. If we combine equation (2) with the equation of motion for $\vec{g}_e$, we obtain

$$\frac{\partial}{\partial t}\left(\vec{g}_e + \vec{g}_{ion}\right) + \nabla_r P_e + e\vec{E}\left(Z\tilde{n}_i - \tilde{n}_e\right) = 0 \tag{3}$$

upon ignoring the quadratic terms in $\vec{v}_d$. We have set the collision term equal to zero because once we have included the ion degrees of freedom, we can invoke momentum conservation.

To eliminate the particle densities from equation (3), we consider the time derivative of the electron number density

$$\begin{aligned}
\left.\frac{\partial \tilde{n}_e}{\partial t}\right|_{coll} &= 2\int \frac{d\vec{p}}{(2\pi\hbar)^3}\left.\frac{\partial f_p}{\partial t}\right|_{coll} \\
&= \frac{\partial \tilde{n}_e}{\partial t} + 2\int \frac{d\vec{p}}{(2\pi\hbar)^3}\left[\nabla_p \epsilon_p \cdot \nabla_r f_p - \nabla_r \epsilon_p \cdot \nabla_p f_p\right] \\
&= \frac{\partial \tilde{n}_e}{\partial t} + \nabla \cdot \vec{J}(r,t)
\end{aligned} \tag{4}$$

where $\vec{J}(\vec{r},t)$ is the particle current

$$\begin{aligned}
\vec{J}(r,t) &= 2\int \frac{d\vec{p}}{(2\pi\hbar)^3}\left(\nabla_p \epsilon_p\right) f_p \\
&\cong \vec{v}_d \tilde{n}_e
\end{aligned} \tag{5}$$

Because the collision terms vanish in an isolated system, we have that

$$\frac{\partial \tilde{n}_e}{\partial t} + \nabla_r \cdot (\vec{v}_d \tilde{n}_e) = 0 \tag{6}$$

In general, $\tilde{n}_e$ is a function of position because it involves an integral over the complete distribution function in the presence of the drift velocity. Nonetheless, we linearize about the equilibrium unperturbed value, n_e, and obtain

$$\frac{\partial \tilde{n}_e}{\partial t} + n_e \nabla_r \cdot \vec{v}_d = 0 \tag{7}$$

and for the ion degrees of freedom,

$$\frac{\partial \tilde{n}_i}{\partial t} + n_i \nabla_r \cdot \vec{u}_{ion} = 0 \tag{8}$$

If the electrons and ions are moving colletively, then $\vec{v}_d \sim \vec{u}_{ion}$. If we compare equations (7) and (8) with the insight that $n_e = Zn_i$, we find that even in the drift-velocity distribution,

$$\frac{\partial \tilde{n}_e}{\partial t} = Z \frac{\partial \tilde{n}_i}{\partial t} \tag{9}$$

or, equivalently, $\tilde{n}_e = Z\tilde{n}_i$. As a consequence, the last term vanishes in equation (3), and we obtain that

$$\frac{\partial}{\partial t} (\vec{g}_e + \vec{g}_{ion}) + \nabla_r P_e = 0 \tag{10}$$

Using the equilibrium form for $\vec{g}$, equation (**??**), and the approximation in equation (5), we reduce the momentum conservation constraint to

$$\frac{\partial}{\partial t} [(m\tilde{n}_e + M\tilde{n}_i) v_d] + \nabla_r P_e = 0 \tag{11}$$

To eliminate the drift-velocity terms, we take the divergence of equation (11) and linearize, keeping only the $\nabla_r v_d$ derivative:

$$(mn_e + Mn_i) \frac{\partial}{\partial t} \nabla_r \cdot \vec{v}_d + \nabla_r^2 P_e = 0 \tag{12}$$

Coupled with the equation for the second time derivative of the electron number density,

$$\frac{\partial^2 \tilde{n}_e}{\partial t^2} + n_e \frac{\partial}{\partial t} (\nabla_r \cdot \vec{v}_d) = 0 \tag{13}$$

we arrive at the equation of motion,

$$\left(m + \frac{M}{Z}\right) \frac{\partial^2 \tilde{n}_e}{\partial t^2} - \nabla_r^2 P_e = 0 \tag{14}$$

In equation (14), we can ignore the elctron mass relative to the ion mass. Let us calculate the gradient of the pressure. We note that $\nabla_r P_e = \partial P_e \partial \tilde{n}_e / \nabla_r \tilde{n}_e$. We showed previously that

$$\frac{\partial P_e}{\partial \tilde{n}_e} = \frac{2\mu_0}{3} = \frac{mv_F^2}{3} \tag{15}$$

The equation of motion for the elctron density, then, becomes

$$\frac{\partial^2 \tilde{n}_e}{\partial t^2} - \frac{Z}{M}\frac{mv_F^2}{3}\nabla_r^2 \tilde{n}_e = 0 \tag{16}$$

In Fourier space, we obtain

$$\left(-\omega^2 + \frac{Zm}{M}\frac{v_F^2}{3}k^2\right)\tilde{n}_e = 0 \tag{17}$$

or,

$$\omega^2 = \frac{Zm}{M}\frac{v_F^2}{3}k^2 = v_{sound}^2 k^2 \tag{18}$$

Whihc is quadratic in the wave vector k. The speed of sound in a metal is, then,

$$v_{sound} = \sqrt{\frac{Zm}{M}\frac{v_F^2}{3}} \tag{19}$$

which is the Bohm-Staver result. It accurately describes the propagation of sound in a metal in the hydrodynamic limit. Plasma oscillations in general cannot be treated in an analogous manner because they are outside ($\omega\tau_{ee} >> 1$) the hydrodynamic limit.

2 Summary

Two key results were derived in this chapter:

1. the form of linear interaction between electrons and phonons, and
2. the Boltzmann tranport theory.

Using the latter, we were able to derive the result we advertised initially in Chapter 7, namely ,that the resisistivity vanishes as T^5 in a metal where collisions with phonons dominate all scattering processes. We derived this result using the relaxation-time approximation. An essential ingredient of the relaxation-time approximation is the drift-velocity ansatz. In this approximation the Fermi surface is translated by an amount proportional to the drift velocity. Electrons in the translated Fermi surface are described by the original Fermi-Dirac distribution with the momentum shifted by $\vec{p} \to \vec{p} - m\vec{v}_d$.

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