Magnetic moment due to orbital motion of electron

If a particle of mass m carrying a charge e is rotating in an elliptical orbit of area A as in Figure, it has a magnetic moment μ_l given by

$$\mu_l = iA = \frac{e}{T}A \tag{1}$$

where T is the time period of revolution.
Areal velocity in a central orbit is given by,

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\phi}{dt} \tag{2}$$

The angular momentum of the charge particle (say electron in this case), is given by,

$$L = mr^2\frac{d\phi}{dt} = constant$$

Dividing both sides by $2m$,

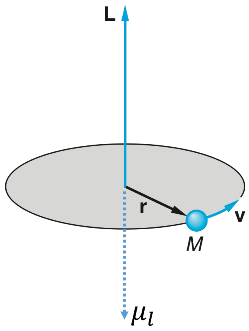

Figure 1: A particle moving in an elliptical orbit has angular momentum L. If the particle has a negative charge, the magnetic moment due to the current is anti-parallel to L.

$$\frac{L}{2m} = \frac{1}{2}r^2\frac{d\phi}{dt} = constant \quad \Rightarrow \frac{L}{2m} = \frac{dA}{dt}\text{(from equation (2))} \quad \Rightarrow dA = \frac{L}{2m}dt$$

$$\Rightarrow A = \frac{L}{2m}T$$

Therefore, from equation (1)

$$\boxed{\mu_l = \frac{e}{2m}L} \tag{3}$$

$$\mu_l = \frac{e}{2m}\sqrt{l(l+1)}\hbar = \frac{e\hbar}{2m}\sqrt{l(l+1)} = \mu_B\sqrt{l(l+1)}$$

where $\mu_B = \frac{e\hbar}{2m}$ is called *Bohr magneton*, l is the orbital quantum number ($l = 0, 1, 2, 3, ...$).

$$\mu_B = \frac{e\hbar}{2m} = 9.27 \times 10^{-24} joules/tesla = 5.79 \times 10^{-5} eV/tesla$$

Since the electronic charge is negative, the magnetic dipole moment vector is directed opposite to that of the angular momentum. The ratio of magnetic dipole moment to angular momentum is called the *gyromagnetic ratio (g)*.

$$g = \frac{\mu_l}{L} = \frac{e}{2m} = 8.8 \times 10^9 C/kg$$

Generally, a system of electrons possessing a total angular momentum J has a magnetic moment μ anti-parallel to J.

$$\mu = -\frac{g\mu_B J}{\hbar} \tag{4}$$

When a magnetic dipole μ is placed in an external magnetic field B, it experiences a torque

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

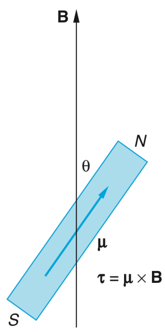

Figure 2: Bar-magnet model of magnetic moment. In an external magnetic field, the moment experiences a torque that tends to align it with the field.

The torque tends to align the dipole moment vector along that of the field. Maximum torque occurs when the angle θ between $\vec{\mu}$ and $\vec{B}$ is 90°. The potential energy of a magnetic dipole, at any angle θ relative to an external magnetic field is

$$U = -\vec{\mu} \cdot \vec{B}$$

An electron spinning about its axis should also behave as a tiny magnet and possess a magnetic dipole moment due to this spin. However, nothing is known about the shape of an electron or the manner in which its charge is distributed. Hence it is impossible to calculate its spin magnetic dipole moment in a manner similar to that used for the orbital motion. In order to obtain agreement with experimental results, the spin magnetic dipole moment μ_s is expressed as

$$\mu_s = \frac{e}{2m} S = \sqrt{s(s+1)}\mu_B = \sqrt{\frac{3}{4}}\mu_B$$

Stern and Gerlach Experiment

The experimental setup is shown in figure. Atoms from an oven are collimated and sent through a magnet whose poles are shaped so that the magnetic field B_z increases slightly with z, while B_x and B_y are constant in the x and y directions,respectively. The atoms then strike a collector plate.

The experiment is based on the behavior of a magnetic dipole in *non-uniform magnetic field*. In a uniform magnetic field B , the dipole experiences a torque that tends to align the dipole parallel to the field. If the dipole moves in such a field in a direction normal to the field, it will trace a straight line path without any deviation. In an inhomogeneous magnetic field, the dipole experiences, in addition, a translatory force. If the atomic magnet flies across such an inhomogeneous magnetic field normal to the field direction, it will be deviated away from its straight path.

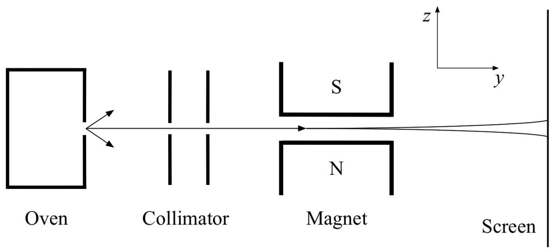

Figure 3: Schematic diagram of experimental setup of Stern-Gerlach Experiment

Let the magnetic field vary along the z-direction, so that the field gradient is dB/dx and is positive. Let the atomic magnet represented by CD in the figure, is of length l and pole strength p.

Let, magnetic field at pole C be B, then, magnetic field at pole D is $B + \frac{dB}{dz} l \cos\theta$

Hence the forces on the two poles is given by pB at pole C and $p\left[B + \frac{dB}{dz} l \cos\theta\right]$ at pole D.

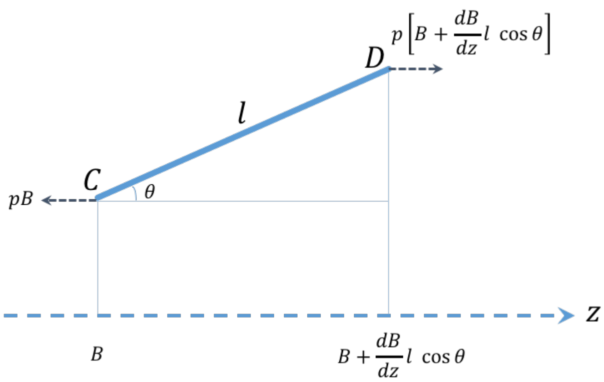

Figure 4: Atomic magnet in a *non-uniform magnetic field.*

Hence the atomic magnet experiences not only a torque but also a translatory force given by

$$F_z = \frac{dB}{dz}\, pl\ \cos\theta$$
$$\Rightarrow F_z = \frac{dB}{dz}\, \mu_s\ \cos\theta$$

Let V be the velocity of the atomic magnet of mass m as it enters the field; L be the length of the path of the atom in the field; t be the time of travel of the atom through the field.

The acceleration given to the atom along the field direction, by the force F_z is

$$acceleration = \frac{F_z}{m};$$

Displacement of the atom along the field direction, on emerging out of the field is,

$$displacement = \frac{1}{2} \times acceleration \times t^2$$
$$= \frac{1}{2} \times \frac{F_z}{m} \times \frac{L^2}{V^2}$$
$$= \frac{1}{2}\frac{dB}{dz}\frac{\mu_s \cos\theta}{m}\frac{L^2}{V^2}$$

if μ is the resolved component of the magnetic moment in the field direction, $\mu = \mu_s \cos\theta$,

$$displacement = \frac{1}{2}\frac{dB}{dz}\frac{\mu}{m}\frac{L^2}{V^2}$$

Knowing *displacement*, $\frac{dB}{dz}$, L, and V, μ can be calculated. Silver atom was used in the original experiment and the magnetic moment μ of the silver atom was found to be one *Bohr magneton* in the direction of the field.

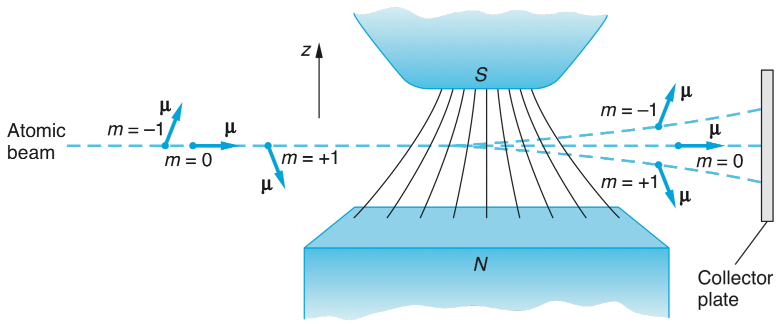

Figure 5: Stern Gerlach Experiment